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A Note on Conifolds

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Abstract

We present the Ricci-flat metric and its Kähler potential on the conifold with the $O(N)$ isometry, whose conical singularity is repaired by the complex quadric surface $Q^{N-2} = SO(N)/SO(N-2) \times U(1)$.

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Introduction. Conformally invariant nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry in two-dimensions describe the superstring in curved space. The target space must be a Ricci-flat Kähler manifold by the requirement of finiteness [1, 2, 3]. In the previous letter [4], we presented the simple derivation of the Ricci-flat metric on the deformed conifold with the $O(N)$ isometry, whose conical singularity is removed by S^{N-1}. It coincides with the Stenzel metric on the cotangent bundle over S^{N-1} [5], and includes the Eguchi-Hanson gravitational instanton [6] and the six-dimensional deformed conifold [7, 8] in the cases of $N = 3$ and $N = 4$, respectively. The metric contains the deformation parameter, and the manifold becomes a conifold when the parameter vanishes.

In this letter, we present the explicit form of the Ricci-flat metric and its Kähler potential on the conifold, whose conical singularity is repaired by the *complex quadric surface* $Q^{N-2} \equiv SO(N)/SO(N-2) \times U(1)$. It contains a resolution parameter b as an integration constant, which controls the size of Q^{N-2}. The limit of $b \to 0$ corresponds to the conifold, which coincides with the singular limit of the deformed conifold. Our manifold can be interpreted as the line bundle over Q^{N-2}. The four-dimensional manifold of $N = 3$ is again the Eguchi-Hanson space, in which the conical singularity is removed by $Q^1 \simeq S^2$. In the case of the six-dimensional manifold of $N = 4$, the conical singularity is repaired by $Q^2 \simeq S^2 \times S^2$, and it gives a way to repair the singularity different from the *deformation* by S^3 [7, 8] or the so-called *small resolution* by S^2 [7, 9].

Definition of the model. $\mathcal{N} = 2$ supersymmetric nonlinear sigma models in two dimensions are described by the chiral superfields $\varphi^\alpha(x, \theta, \bar{\theta})$ and the Kähler potential $\mathcal{K}(\varphi, \varphi^*)$ [10]. The Lagrangian is given by $\mathcal{L} = \int d^4\theta\, \mathcal{K} = g_{\alpha\beta^*}(\varphi, \varphi^*)\partial_\mu\varphi^\alpha\partial^\mu\varphi^{*\beta} + \cdots$, where the Kähler metric is defined by $g_{\alpha\beta^*} = \partial_\alpha\partial_{\beta^*}\mathcal{K}$ with $\partial_\alpha = \partial/\partial\varphi^\alpha$ and $\partial_{\alpha^*} = \partial/\partial\varphi^{*\alpha}$. (Here we have used the same letters for chiral superfields and their components.)

First, we prepare chiral superfields $\phi^A(x, \theta, \bar{\theta})$ $(A = 1, 2, \cdots, N;\ N \geq 3)$, constituting the vector $\vec{\phi}(x, \theta, \bar{\theta})$ of $O(N)$. We define the $O(N)$ symmetric target space by imposing the constraint

$$\sum_{A=1}^{N} (\phi^A)^2 \ = \ 0\,. \tag{1}$$

This constraint defines the conifold with the real dimension $2N - 2$. We can rewrite this by an unitary transformation as

$$\vec{\phi}^T J \vec{\phi} \ = \ 0. \tag{2}$$

Here J is the rank-2 invariant tensor of $O(N)$, which we take as

$$J \ = \ \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{1}_{N-2} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}\,. \tag{3}$$

where $\mathbf{1}_{N-2}$ is the $(N-2)\times(N-2)$ unit matrix.

Introducing an auxiliary chiral superfield $\phi_0(x,\theta,\bar\theta)$, we can give the $O(N)$ symmetric Lagrangian by

$$\mathcal{L} \;=\; \int d^4\theta\, \mathcal{K}(X) + \left(\int d^2\theta\ \phi_0 \vec\phi^T J \vec\phi + \text{c.c.} \right) . \tag{4}$$

Here, $X(x,\theta,\bar\theta)$ is the $O(N)$-invariant real superfield, defined by

$$X \;\equiv\; \sum_{A=1}^{N} \phi^{\dagger A}\phi^A , \tag{5}$$

and $\mathcal{K}(X)$ is an arbitrary function of X. The symmetry of the Lagrangian (4) is $G=O(N)\times U(1)$, assigning the $U(1)$ charges of ϕ^A and ϕ_0, 1 and -2, respectively. By the integration over the auxiliary field ϕ_0, we obtain the constraint (2), which can be immediately solved as

$$\vec\phi \;=\; \sigma \begin{pmatrix} 1 \\ z^i \\ -\frac{1}{2}(z^i)^2 \end{pmatrix} , \tag{6}$$

where the summation over the repeated indices is implied. Here $\sigma(x,\theta,\bar\theta)$ and $z^i(x,\theta,\bar\theta)$ $(i=1,2,\cdots,N-2)$ are chiral superfields, with the $U(1)$ charges 1 and 0, respectively. Scalar components of these superfields parameterize the target space, and the symmetry G acts on those fields as a holomorphic isometry. The invariant X becomes

$$X \;=\; |\sigma|^2 \left[1 + |z^i|^2 + \frac{1}{4}(z^i)^2(z^{*j})^2 \right] \;\equiv\; |\sigma|^2 Z . \tag{7}$$

Note that the constraint (1) or (2) is invariant under the complex extension of the symmetry G. Using this, any point $\vec\phi$ on the manifold can be transformed to $\langle \vec\phi^T \rangle = (1,0,\cdots,0)$, which can be interpreted as the vacuum expectation value. From this, we find the symmetry G is spontaneously broken down to $H=O(N-2)\times U(1)$. Hence there appear the Nambu-Goldstone bosons, parameterizing $G/H \simeq SO(N)/SO(N-2)$. The whole target manifold can be locally regarded as $\mathbf{R}\times SO(N)/SO(N-2) \simeq \mathbf{R}\times S^{N-1}\times S^{N-2}$.

Ricci-flat Condition and Its Solution. We would like to determine the function $\mathcal{K}(X)$, imposing the Ricci-flat condition on the manifold. We use the *same* letters for superfields *and* their lowest components from now on. The Kähler metric is

$$g_{\alpha\beta^*}(\varphi,\varphi^*) \;=\; \partial^2\mathcal{K}(X)\partial\varphi^\alpha\partial\varphi^{*\beta} \;=\; \frac{d^2\mathcal{K}}{dX^2}\frac{\partial X}{\partial\varphi^\alpha}\frac{\partial X}{\partial\varphi^{*\beta}} + \frac{d\mathcal{K}}{dX}\frac{\partial^2 X}{\partial\varphi^\alpha\partial\varphi^{*\beta}} , \tag{8}$$

where $\varphi^\alpha \equiv (\sigma, z^i)$. The Ricci form is given by $(Ric)_{\alpha\beta^*} = -\partial_\alpha\partial_{\beta^*}\log\det g_{\gamma\delta^*}$, and the Ricci-flat condition $(Ric)_{\alpha\beta^*}=0$ implies $\det g_{\alpha\beta^*} = (\text{constant})\times|F|^2$, with F being a holomorphic function.

This is a partial differential equation, which is difficult to solve in general. The determinant $\det g_{\alpha\beta^*}$ can be calculated as

$$\det g_{\alpha\beta^*} \ = \ \frac{X}{|\sigma|^2}\Big(X\frac{d^2\mathcal{K}}{dX^2}+\frac{d\mathcal{K}}{dX}\Big)\Big(|\sigma|^2\frac{d\mathcal{K}}{dX}\Big)^{N-2}\cdot\det(\partial_i\partial_{j^*}Z-Z^{-1}\partial_iZ\partial_{j^*}Z)\ , \tag{9}$$

where ∂_i denotes the differentiation with respect to z^i: $\partial_i Z = z^{*i}+\frac{1}{2}z^i(z^{*j})^2$ and $\partial_i\partial_{j^*}Z=\delta_{ij}+z^iz^{*j}$. Using the complex extension of the isotropy H, $SO(N-2,\mathbf{C})$, we can choose a point labeled by $z^1\neq 0$ and $z^m=0$ ($m=2,3,\cdots,N-2$), without loss of generality. At that point, we find

$$\det(\partial_i\partial_{j^*}Z-Z^{-1}\partial_iZ\partial_{j^*}Z) \ = \ \det\delta_{ij} \ = \ 1\ , \quad X \ = \ |\sigma|^2\left(1+|z^1|^22\right)^2\ , \tag{10}$$

and then obtain

$$\det g_{\alpha\beta^*} \ = \ |\sigma|^{2N-6}\Big(\frac{d\mathcal{K}}{dX}\Big)^{N-2}\Big(X^2\frac{d^2\mathcal{K}}{dX^2}+X\frac{d\mathcal{K}}{dX}\Big)\ . \tag{11}$$

Therefore, the Ricci-flat condition becomes an ordinary differential equation:

$$\Big(\frac{d\mathcal{K}}{dX}\Big)^{N-2}\Big(X^2\frac{d^2\mathcal{K}}{dX^2}+X\frac{d\mathcal{K}}{dX}\Big) \ = \ \frac{1}{N-1}X^2\frac{d}{dX}\left[\Big(\frac{d\mathcal{K}}{dX}\Big)^{N-1}\right]+X\Big(\frac{d\mathcal{K}}{dX}\Big)^{N-1} \ \equiv \ c\ , \tag{12}$$

where c is a constant. This can be immediately solved as

$$\frac{d\mathcal{K}}{dX} \ = \ (\lambda X^{N-2}+b)^{\frac{1}{N-1}}X\ , \tag{13}$$

where λ is a constant related to c and N, and b is an integration constant. We impose $b\geq 0$ and $\lambda>0$ in order that the Kähler potential is real.

The solution (13) is sufficient to obtain the Ricci-flat metric using (8), but we can calculate its Kähler potential by integrating (13):

$$\mathcal{K}(X) \ = \ \frac{N-1}{N-2}\left[(\lambda X^{N-2}+b)^{\frac{1}{N-1}}+b^{\frac{1}{N-1}}\cdot I\Big(b^{\frac{1}{1-N}}\left(\lambda X^{N-2}+b\right)^{\frac{1}{N-1}};N-1\Big)\right]\ , \tag{14}$$

where the function $I(y;n=N-1)$ is defined by

$$\begin{aligned} I(y;n) \ \equiv \ \int^y\frac{dt}{t^n-1} \ &= \ \frac{1}{n}\Big[\log\left(y-1\right)-\frac{1+(-1)^n}{2}\log\left(y+1\right)\Big] \\ &+\frac{1}{n}\sum_{r=1}^{[\frac{n-1}{2}]}\cos\frac{2r\pi}{n}\cdot\log\Big(y^2-2y\cos\frac{2r\pi}{n}+1\Big) \\ &+\frac{2}{n}\sum_{r=1}^{[\frac{n-1}{2}]}\sin\frac{2r\pi}{n}\cdot\arctan\Big[\frac{\cos(2r\pi/n)-y}{\sin(2r\pi/n)}\Big]\ . \end{aligned} \tag{15}$$

If we set $b=0$ in (14), it becomes the Kähler potential of the conifold, which coincides with the one of the singular limit of the deformed conifold [4].

Ricci-flat Metric. Using Eqs. (8) and (13), the components of the Ricci-flat metric can be calculated, to give

$$
\begin{aligned}
g_{\sigma\sigma^*} &= \lambda\Big(\frac{N-2}{N-1}\Big)\big(\lambda X^{N-2}+b\big)^{\frac{2-N}{N-1}} X^{N-2}|\sigma|^{-2} , && (16a)\\
g_{\sigma j^*} &= \lambda\Big(\frac{N-2}{N-1}\Big)\big(\lambda X^{N-2}+b\big)^{\frac{2-N}{N-1}} X^{N-3}\sigma^*\partial_{j^*}Z , && (16b)\\
g_{ij^*} &= \lambda\Big(\frac{N-2}{N-1}\Big)\big(\lambda X^{N-2}+b\big)^{\frac{2-N}{N-1}} X^{N-4}|\sigma|^4\partial_i Z\partial_{j^*}Z \\
&\quad + \big(\lambda X^{N-2}+b\big)^{\frac{1}{N-1}}(Z^{-1}\partial_i\partial_{j^*}Z - Z^{-2}\partial_i Z\partial_{j^*}Z) . && (16c)
\end{aligned}
$$

This Kähler metric is singular at the surface defined by $\sigma = 0$: $g_{\sigma\sigma^*}|_{\sigma=0} = 0$. However this is just a coordinate singularity of the coordinate system $(\sigma,\, z^i)$. To find regular coordinates, let us perform a coordinate transformation

$$
\rho \equiv \frac{\sigma^{N-2}}{N-2} , \qquad (17)
$$

with z^i being unchanged. The components of the Kähler metric in the new coordinates (ρ, z^i) are

$$
\begin{aligned}
g_{\rho\rho^*} &= \lambda\Big(\frac{N-2}{N-1}\Big)\big(\lambda X^{N-2}+b\big)^{\frac{2-N}{N-1}} Z^{N-2} , && (18a)\\
g_{\rho j^*} &= \lambda\frac{(N-2)^2}{N-1}\big(\lambda X^{N-2}+b\big)^{\frac{2-N}{N-1}} \rho^* Z^{N-3}\partial_{j^*}Z , && (18b)\\
g_{ij^*} &= \lambda\frac{(N-2)^3}{N-1}\big(\lambda X^{N-2}+b\big)^{\frac{2-N}{N-1}} |\rho|^2 Z^{N-4}\partial_i Z\partial_{j^*}Z \\
&\quad + \big(\lambda X^{N-2}+b\big)^{\frac{1}{N-1}}(Z^{-1}\partial_i\partial_{j^*}Z - Z^{-2}\partial_i Z\partial_{j^*}Z) , && (18c)
\end{aligned}
$$

where $X = |(N-2)\rho|^{2N-2}Z$. These are non-singular at the surface of $\rho = 0$, corresponding to $\sigma = 0$, as long as the integration constant b takes a non-zero value. In the limit of $b \to 0$, the manifold becomes the conifold and the metric (18) becomes singular at $\rho = 0$. So we can regard this constant b as a *resolution* parameter of the conical singularity. The coordinate singularity in the coordinates (σ, z^i) is due to the identification of (17) as in the Calabi metric on the line bundle over $\mathbf{C}P^{N-1}$ [11].

The metric of the $\rho = 0$ surface itself ($d\rho = 0$) is

$$
g_{ij^*}(z, z^*) = b^{\frac{1}{N-1}}(Z^{-1}\partial_i\partial_{j^*}Z - Z^{-2}\partial_i Z\partial_{j^*}Z). \qquad (19)
$$

This define a Kähler submanifold whose Kähler potential is given by

$$
\mathcal{K}(z, z^*) = b^{\frac{1}{N-1}} \log\left[1 + |z^i|^2 + \frac{1}{4}(z^i)^2(z^{*j})^2\right] = b^{\frac{1}{N-1}}\log Z , \qquad (20)
$$

which is the Kähler potential of the complex quadric surface $Q^{N-2} = SO(N)/SO(N-2)\times U(1)$ [12]–[15]. Therefore we have found that the conical singularity is resolved by Q^{N-2} of the radius $b^{\frac{1}{2(N-1)}}$.

The manifold can be interpreted as the line bundle over Q^{N-2}. In fact it was proved in [16] that there exists a Ricci-flat Kähler metric on the line bundle over any Einstein manifold.

Examples. Let us give the more concrete expressions for the $N = 3$ and $N = 4$ cases. For the four-dimensional manifold of $N = 3$, the Kähler potential (14) becomes

$$\mathcal{K}(X) \;=\; 2\sqrt{\lambda X + b} + \sqrt{b}\log\left(\frac{\sqrt{\lambda X + b} - \sqrt{b}}{\sqrt{\lambda X + b} + \sqrt{b}}\right) . \tag{21a}$$

Defining $\varrho^4 = 4(\lambda X + b)$ and $a^4 = 4b$, we find that this is the Kähler potential [17] of the Eguchi-Hanson gravitational instanton [6]:

$$\mathcal{K} \;=\; \varrho^2 + \frac{a^2}{2}\log\left(\frac{\varrho^2 - a^2}{\varrho^2 + a^2}\right) . \tag{22}$$

The singularity at the apex of the conifold is repaired by $Q^1 \simeq S^2$, and the isometry is $SO(3) \times U(1) \simeq U(2)$.

The Kähler potential (14) in the six-dimensional manifold of $N = 4$ is

$$\begin{aligned}\mathcal{K}(X) \;=\; & \frac{3}{2}(\lambda X^2 + b)^{1/3} + \frac{b^{1/3}}{4}\log\left[\frac{\{(\lambda X^2 + b)^{1/3} - b^{1/3}\}^3}{\lambda X^2}\right] \\ & - \frac{\sqrt{3}b^{1/3}}{2}\arctan\left[\frac{2(\lambda X^2 + b)^{1/3} + b^{1/3}}{\sqrt{3}b^{1/3}}\right] .\end{aligned} \tag{23}$$

The metric in the coordinates $(\rho,\, z^1,\, z^2)$ is represented as follows:

$$g_{\rho\rho^*} \;=\; \frac{2\lambda}{3}\frac{Z^2}{(\lambda X^2 + b)^{2/3}} , \quad g_{\rho j^*} \;=\; \frac{4\lambda}{3}\frac{\rho^* Z\partial_{j^*}Z}{(\lambda X^2 + b)^{2/3}} , \tag{24a}$$

$$g_{ij^*} \;=\; \frac{8\lambda}{3}\frac{|\rho|^2\partial_i Z\partial_{j^*}Z}{(\lambda X^2 + b)^{2/3}} + (\lambda X^2 + b)^{1/3}(Z^{-1}\partial_i\partial_{j^*}Z - Z^{-2}\partial_i Z\partial_{j^*}Z) . \tag{24b}$$

The isometry of this manifold is $SO(4) \times U(1) \simeq SU(2) \times SU(2) \times U(1)$. The singularity at the apex of the conifold is repaired by $Q^2 \simeq S^2 \times S^2$ (the radii of these two S^2 coincide). This way of repairing of the conical singularity is different from either the *deformation* by S^3 [7, 8] or the *small resolution* by S^2 [7, 9] known in the six-dimensional conifold.

Discussions. We can obtain the Kähler potential (20) of $Q^{N-2} = SO(N)/SO(N-2) \times U(1)$ directly, by gauging the $U(1)$ part of the isometry G and performing the integration over gauge superfields. (This is known as Kähler quotient [18], and actually hold for an arbitrary Kähler potential $\mathcal{K}(X)$ [13].) Replacing the base manifold Q^{N-2} by other compact manifolds in [12, 14], we can construct other Ricci-flat Kähler manifolds, whose conical singularity is repaired by those base manifolds [19]. Since non-perturbative effects of Q^N was investigated using the large-N method in [15], the large-N limit of the conifold is also interesting. The investigation of super-conformal field theories corresponding to our manifolds is an interesting task.

After the completion of this work we were informed that the six-dimensional manifold in the $N = 4$ case is known in refs. [20, 21].

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References

[1] L. Alvarez-Gaumé, Nucl. Phys. **B184** (1981) 180.
L. Alvarez-Gaumé and P. Ginsparg, Comm. Math. Phys. **102** (1985) 311.

[2] M. T. Grisaru, A. E. M. van de Ven and D. Zanon, Phys. Lett. **173B** (1986) 423; Nucl. Phys. **B277** (1986) 388; Nucl. Phys. **B277** (1986) 409.

[3] D. Nemeschansky and A. Sen, Phys. Lett. **178B** (1986) 365.

[4] K. Higashijima, T. Kimura and M. Nitta, `hep-th/0104184`, to appear in Phys. Lett. **B**.

[5] M. B. Stenzel, Manuscripta Math. **80** (1993) 151.
M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope, `hep-th/0012011`.

[6] T. Eguchi and A. J. Hanson, Phys. Lett. **74B** (1978) 249.

[7] P. Candelas and X. C. de la Ossa, Nucl. Phys. **B342** (1990) 246.

[8] K. Ohta and T. Yokono, JHEP **0002** (2000) 023, `hep-th/9912266`.

[9] L. A. Pando Zayas and A. A. Tseytlin, JHEP **0011** (2000) 028, `hep-th/0010088`.

[10] B. Zumino, Phys. Lett. **87B** (1979) 203.

[11] E. Calabi, Ann. Scient. Ec. Norm. Sup. **12** (1979) 269.

[12] K. Higashijima and M. Nitta, Prog. Theor. Phys. **103** (2000) 635, `hep-th/9911139`.

[13] K. Higashijima and M. Nitta, Prog. Theor. Phys. **103** (2000) 833, `hep-th/9911225`.

[14] K. Higashijima and M. Nitta, Proceedings of Confinement 2000 (World Scientific, 2001) 279, `hep-th/0006025`; Proceedings of ICHEP 2000 (World Scientific, 2001) 1368, `hep-th/0008240`.

[15] K. Higashijima, T. Kimura, M. Nitta and M. Tsuzuki, Prog. Theor. Phys. **105** (2001) 261, `hep-th/0010272`.

[16] D. N. Page and C. N. Pope, Class. Quantum Grav. **4** (1987) 213.

[17] G. W. Gibbons and C. N. Pope, Comm. Math. Phys. **66** (1979) 267.

[18] U. Lindström and M. Roček, Nucl. Phys. **B222** (1983) 285.
N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Comm. Math. Phys. **108** (1987) 535.

[19] K. Higashijima, T. Kimura and M. Nitta, `hep-th/0108084`.

[20] T. Eguchi and M. Naka, unpublished (2000).

[21] L. A. Pando Zayas and A. A. Tseytlin, Phys. Rev. **D63** (2001) 086006, `hep-th/0101043`.